Lecture 12

Tuesday, 6 October 2026 11:32 PM

Social Welfare Fn: $F : L(A)^n \to L(A)$

$P = (>_1, \ldots, >_n) \to >$

Axioms:

1. $\forall i$ $a >_i b \Rightarrow a > b$ (Unanimity)
2. $\exists i$: for every profile P, (Dictatorship)
   $F(P) = >_i$
3. Consider P & P' s.t. $\forall i$ $(a >_i b) \Leftrightarrow (a >'_i b)$
   Then $(a > b) \Leftrightarrow (a >' b)$
   (Independence of Irrelevant Alternatives)
   i.e., if every agent maintains the same order btw a & b, the o/p does not change either.

Example: Borda violates IIA.

P: 3x $a > b > c$ P': 3x $a > c > b$

2x $b > c > a$ 2x $b > a > c$

Scores: $a:6$ $b:7$ $c:2$ $a:8$ $b:4$ $c:3$

$b > a > c$ $a > b > c$

Arrow's Theorem (1951): For $m \geq 3$, F satisfying unanimity & IIA $\Rightarrow$ F is dictatorial.

W.L.O.G. prove for $m = 3$, i.e. $A = \{a, b, c\}$. Easily extended.

Proof: Assume F is unanimous & IIA.

Claim 1: If $\forall i$, $\text{rank}_i(b) \in \{1, 3\}$, then $\text{rank}_*(b) \in \{1, 3\}$ (i.e., in the output linear order)

Proof: Let P be a profile where (1) $\forall i$ $\text{rank}_i(b) \in \{1, 3\}$ & (11) $F(P) = a > b > c$. Now for each voter i, move c's ranking if needed so that $c >_i a$. Note that this can be done w/o affecting the relative ranking of a & c wrt b. Call this profile P'.

Then in P', $\forall i$, $c >_i a$, so by unanimity, $F(P')$ puts c before a.

But in P & P': $\forall i$ $(c >'_i b) \Leftrightarrow (c >_i b)$, true by IIA $b >'_* c$

& $\forall i$ $(a >_i b) \Leftrightarrow (a >'_i b)$ true by IIA $a >'_* b$

This gives a contradiction.

Step 2: Identify "possible" dictator.

Let $P^0 = \forall i$, $a >_i c >_i b$

$P^1 = b >_1 a >_1 c$

& $\forall i \neq 1$, $a >_i c >_i b$

...

$P^k = \forall i \leq k$, $b >_i a >_i c$

$\forall i > k$, $a >_i c >_i b$

$P^n = \forall i$, $b >_i a >_i c$

Now by the claim, $\forall i$, $F(P^i)$ puts b either first or last

by unanimity, $F(P^0)$ puts b last

$F(P^n)$ puts b first.

Let ℓ be s.t. $F(P^{\ell-1})$ puts b last, $F(P^\ell)$ puts b first. — Fix $P^{\ell-1}$, P^ℓ as here.

Will show that voter ℓ is a dictator, i.e., $\forall P$, $F(P) = >_\ell$

Step 3:

Claim: Let Q be a profile where $a >_\ell c$. Then $F(Q)$ puts a before c.

Proof: Consider Q':

$\forall i < \ell$: a vs c as in Q, b last

ℓ : $a >_\ell b >_\ell c$

$\forall i > \ell$: a vs c as in Q, b first.

① $F(Q') = a > b > c$

$a > b$: consider Q' & $P^{\ell-1}$

$\forall i$, relative position of a & b are the same in both.

$F(P^{\ell-1})$ puts b last, so $F(Q')$ puts $a > b$

$b > c$: consider Q' & P^ℓ

$\forall i$, relative posns. of b & c are the same in both.

$F(P^\ell)$ puts b first, so $F(Q')$ puts $b > c$

Thus $F(Q') = a > b > c$.

Finally, Q & Q' have the same relative posns. for a & c for each voter. Hence $F(Q)$ puts $a > c$. ∎

Step 4:

Claim: $\exists$ a voter ℓ' s.t. for every profile Q, $F(Q)$ puts a before b iff $a >_{\ell'} b$ in Q.

(by following Steps 1 ... 3, & fixing c in place of b)

If $\ell = \ell'$ we are done, voter ℓ is a dictator.

Suppose $\ell \neq \ell'$.

In $P^{\ell-1}$ & P^ℓ:

ℓ''s relative posn of a vs. b did not change.

But $F(P^{\ell-1})$ put b last, $F(P^\ell)$ put b first.

Contradiction. ∎